The University of Melbourne School Mathematics Competition, 2023

INTERMEDIATE DIVISION

Time allowed: Three hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

(1) *Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*

(2) *The* ***seven*** *questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

(3) *It may be necessary to spend considerable time on a problem before any real progress is made.*

(4) *You may need to do rough work but you should then write out your final solution neatly, stating your arguments carefully.*

(5) *Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

Textbooks, electronic calculators and computers are ***NOT*** *allowed. Otherwise normal examination conditions apply.*

1. Let m be a positive integer, such that $2m + 1$ is a perfect square. Show that $m + 1$ is the sum of two successive perfect squares.

2. Consider the arithmetic progression 1, 4, 7, 10, ..., 100. Let A be any set of twenty distinct numbers chosen from this arithmetic progression. Prove that there must be two distinct numbers in A that sum to 104.

3. Two vertical posts, one of height h_1 and the other of height h_2 stand on level ground, a distance l apart. A supporting wire is stretched from the top of each pole to the base of the other. Find the height above ground h_3 of the point at which the two wires cross.

4. If you ask n people what their birthdate is (ignoring the year), what is the minimal number of people you must ask to have at least a 50% chance of finding someone with the same birthdate as you? You can ignore leap years, so assume 365 days in a year. You can express your result as the solution of an equation, without needing to solve that equation.

5. Let a, b, c, d be positive integers satisfying $ab + cd = 34$, $ac + bd = 46$ and $ad + bc = 31$. Find $a + b + c + d$. Then find all solution sets (a, b, c, d).

6. From a point P on the circumference of a circle of radius r, a tangent line is drawn to a point T, where the length of the line PT is 10 units. The shortest distance from T to the circumference of the circle is 5 units. A straight line is drawn from T cutting the circle at point X and then further at point Y. The length of TX is 7.5 units. Find (a) the radius r of the circle, and (b) the length XY.

7. Find all real solutions of the equation

$$x^3 + 4x^2 + 8x + \frac{1}{x^3} + \frac{4}{x^2} + \frac{8}{x} = 140.$$