The University of Melbourne School Mathematics Competition, 2023

SENIOR DIVISION

Time allowed: Three hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

(1) *Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*

(2) *The* ***seven*** *questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

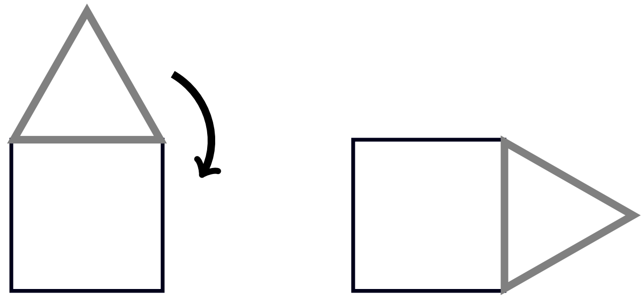
(3) *It may be necessary to spend considerable time on a problem before any real progress is made.*

(4) *You may need to do rough work but you should then write out your final solution neatly, stating your arguments carefully.*

(5) *Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

Textbooks, electronic calculators and computers are ***NOT*** *allowed. Otherwise normal examination conditions apply.*

1. An equilateral triangle of side length 1 meets a square of of side length 1 along an edge as in the diagram. Rotate the triangle around the square, so that the triangle is always in contact with the square and no sliding occurs. The second picture shows the triangle at a later time. If we continue to rotate the triangle until it returns to its original position, how many full rotations around its own centre has the triangle turned? The solution is not necesarily an integer.



2. Tickets to an escape room cost \$40 for adults, \$30 for pensioners and unemployed and \$20 for children. In one day 100 tickets are sold for a total of \$3100. How many of each ticket is sold if an equal amount of money is received for two of the three ticket types?

3. A square target of size 30cm $\times$ 30cm consists of 3×3 smaller 10cm $\times$ 10cm squares with scores as shown in the diagram below. When Enzo aims a dart at a point in the target, his aim is not perfect and the dart lands with uniform probability anywhere inside a circle of radius 10cm centred at the point where Enzo aims. If Enzo always aims at the centre of the target, prove that after three throws his expected total score is greater than 12.

1	3	1
3	9	3
1	3	1

4. Consider the set of all positive integers with either 2022 or 2023 digits and such that any two neighbouring digits are distinct. For example, the set contains

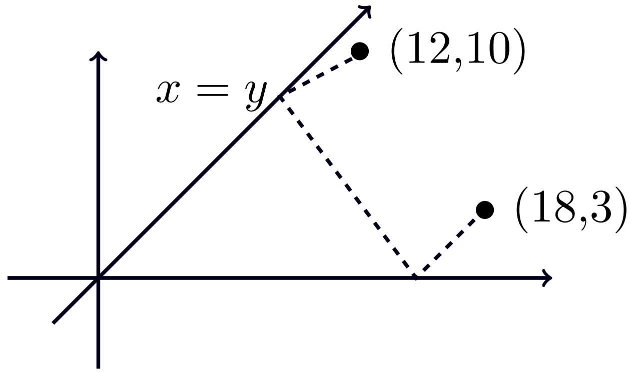
1010101..., 123456123456...

but it does not contain the following in which the same digit appears next to itself:

22222...., 11234...

Find the average of all of the numbers in this set.

5. Find the length of the shortest path consisting of three line segments joined end to end which travels from the point $(12, 10)$ in the cartesian plane to the line $x = y$, then to the x-axis, then to the point $(18, 3)$, as in the diagram.



6. What is the maximum possible area of a right angle triangle with the property that the sum of its side lengths, i.e. its perimeter, is equal to the sum of the squares of its side lengths?

7. A set of three distinct positive integers $\{a, b, c\}$ is defined to be a *tame triple* if $a + b + c = 17$. Define a set S of positive integers to be *tame* if each element of S is contained in *exactly two* tame triples that are subsets of S. For example, the set $\{1, 2, 3, ..., 17\}$ is not tame since 1 is contained in more than two tame triples $\{1, 2, 14\}, \{1, 3, 13\}, \{1, 4, 12\}, ...$ (and 17 is not contained in a tame triple). How many elements must a non-empty tame set S contain?