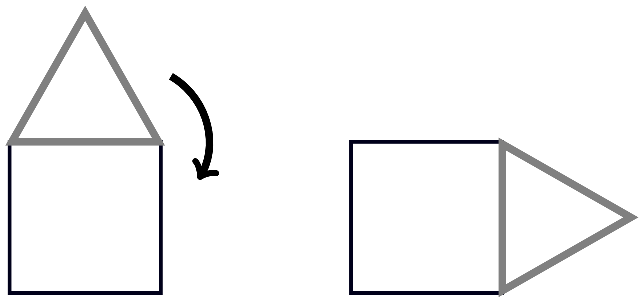
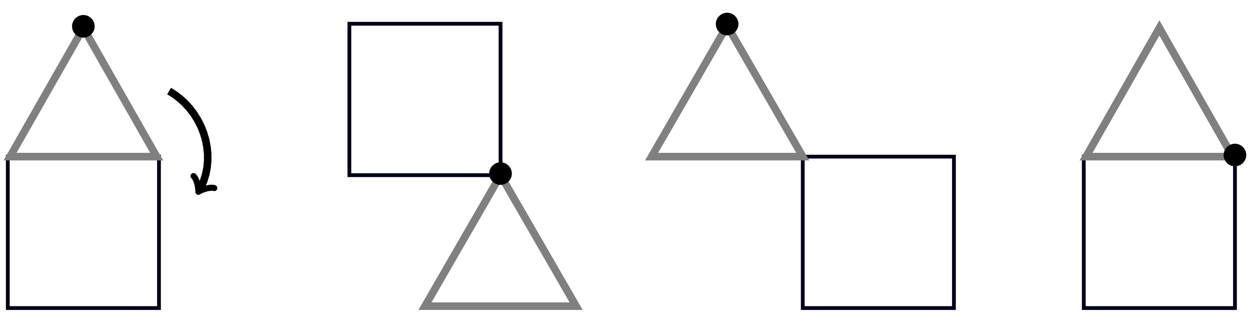
The University of Melbourne School Mathematics Competition, 2023

SENIOR DIVISION
Solutions

1. An equilateral triangle of side length 1 meets a square of of side length 1 along an edge as in the diagram. Rotate the triangle around the square, so that the triangle is always in contact with the square and no sliding occurs. The second picture shows the triangle at a later time. If we continue to rotate the triangle until it returns to its original position, how many full rotations around its own centre has the triangle turned? The solution is not necesarily an integer.



Solution: We can understand rotations by marking the top vertex of the triangle.



The diagrams show two full rotations of the triangle and it needs to rotate a further 1/3 to return to its original position. Hence it rotates $2\frac{1}{3}$ times around its centre point.

2. Tickets to an escape room cost \$40 for adults, \$30 for pensioners and unemployed and \$20 for children. In one day 100 tickets are sold for a total of \$3100. How many of each ticket is sold if an equal amount of money is received for two of the three ticket types?

Solution: Let A = number of adult priced tickets sold, P = number of pensioner and unemployed priced tickets sold, C = number of children priced tickets sold. Then

(1) $$4A + 3P + 2C = 310$$

(2) $$A + P + C = 100$$

Take (1) $- 3 \times$ (2) to get

$$A - C = 10 \Rightarrow A = 10 + C \overset{(2)}{\Rightarrow} P = 90 - 2C$$

Hence

$$(A, P, C) = (10 + C, 90 - 2C, C)$$

and the money (divided by 10) received is

$$(4A, 3P, 2C) = (40 + 4C, 270 - 6C, 2C)$$

Since $40 + 4C > 2C$ there are 2 cases:

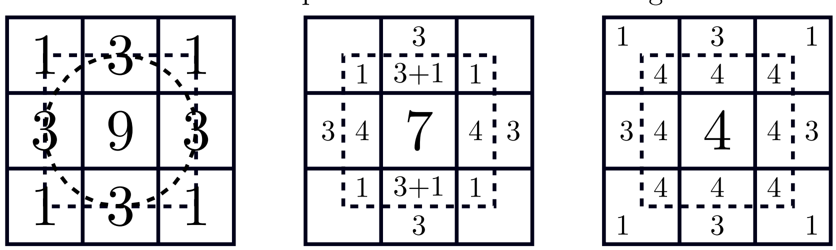
(1) $40 + 4C = 270 - 6C \Rightarrow 10C = 270 - 40 \Rightarrow C = 23 \Rightarrow (A, P, C) = (33, 44, 23)$
(2) $270 - 6C = 2C \Rightarrow C = 270/8$ not possible

Hence $(A, P, C) = (33, 44, 23)$.

3. A square target of size 30cm × 30cm consists of 3 × 3 smaller 10cm × 10cm squares with scores as shown in the diagram below. When Enzo aims a dart at a point in the target, his aim is not perfect and the dart lands with uniform probability anywhere inside a circle of radius 10cm centred at the point where Enzo aims. If Enzo always aims at the centre of the target, prove that after three throws his expected total score is greater than 12.

1	3	1
3	9	3
1	3	1

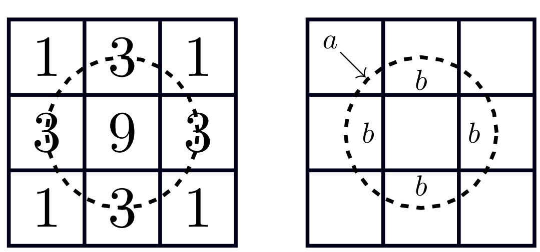
Solution 1: The circle is contained in a square as shown in the left figure below.



Redistribute part of the score of 9 to the neighbouring regions. Add 1 to half of each of the regions with score 3 by removing $4 \times 1/2 = 2$ from the score of 9 as shown in the middle figure. Add 3 to quarter of each of the regions with score 1 by removing $4 \times 3/4 = 3$ from the score of 7 as shown in the right figure.

We see that the area of $9 \times 10 \times 10$ can be resdistributed with some over, so that each region within the score receives a score of 4. Hence the expected value of a single dart throw is strictly greater than 4, the expected total value of three dart throws is strictly greater than 12.

Solution 2: One can also calculate the probability more accurately as follows. Denote by a the area of the intersection of the interior of the circle with any single region of score 1, as shown in the right diagram. Denote by b the area of the intersection of the interior of the circle with any single region of score 3.



The expected value of a single throw of a dart is

$$\frac{4a + 12b + 9 \times 10 \times 10}{100\pi}.$$

The values of a and b can be determined from the following two equations. The area of the circle is given by

$$4a + 4b + 10 \times 10 = 100\pi$$

and the area of a 1/3 wedge of the circle minus the area of an isosceles triangle of angle $2\pi/3$ and side length 1 yields

$$2a + b = (\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3}) \times 100$$

We find that the expected value of one throw is

$$\frac{7}{3} + 2 \times \frac{\sqrt{3}+2}{\pi} > \frac{7}{3} + 2 > 4.$$

4. Consider the set of all positive integers with either 2022 or 2023 digits and such that any two neighbouring digits are distinct. For example, the set contains

$$1010101..., \quad 123456123456...$$

but it does not contain the following in which the same digit appears next to itself:

$$22222...., \quad 11234...$$

Find the average of all of the numbers in this set.

Solution: Denote by D the set of all positive integers with either 2022 or 2023 digits and such that any two neighbouring digits are distinct.

Let $N = 999....99$ consist of 2023 digits consisting only of 9s. If $n \in D$ then $N - n \in D$ since the property that any two neighbouring digits are distinct is preserved. Note that if $n \in D$ is a 2022 digit number then $N - n$ is a 2023 digit number beginning with the digit 9 and conversely if n is a 2023 digit number beginning with the digit 9 then $N - n$ is a 2022 digit number. Essentially we can think of a 2022 digit number as beginning with 0 and any two neighbouring digits remain distinct because the next digit is non-zero.

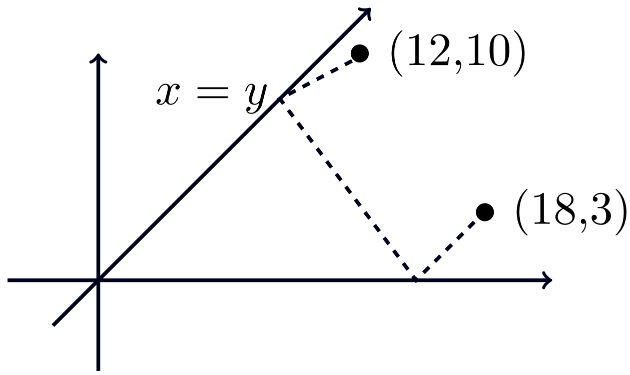
Pair the numbers n and $N - n$ for each $n \in D$. The average can be obtained by redistributing some of the larger number with the smaller number to obtain two copies of

$$\tfrac{1}{2}(n + N - n) = \tfrac{1}{2}N.$$

Since the same number $\frac{1}{2}N$ is obtained twice from each pair, this is the average of all of the numbers in the set. More specifically, the average is

$$\tfrac{1}{2}N = \tfrac{1}{2}(10^{2023} - 1) = 5 \times 10^{2022} - 0.5 = 4\overbrace{9999...99}^{2022\ 9\text{s}}.5$$

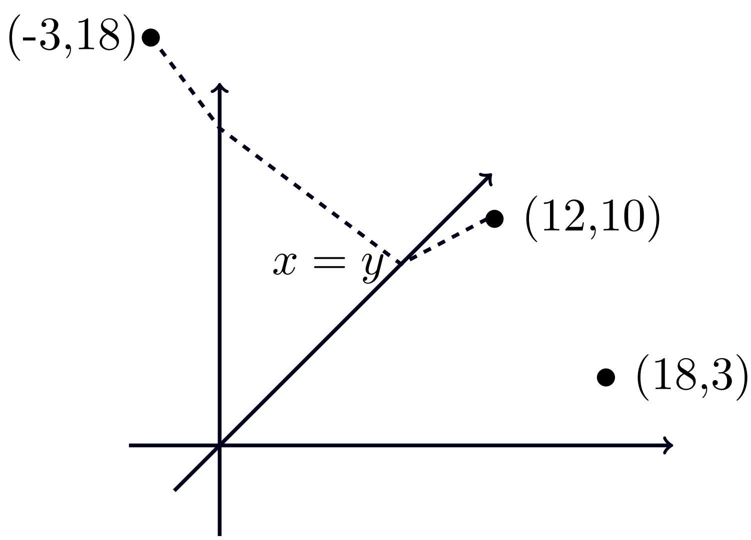
5. Find the length of the shortest path consisting of three line segments joined end to end which travels from the point $(12, 10)$ in the cartesian plane to the line $x = y$, then to the x-axis, then to the point $(18, 3)$, as in the diagram.



Solution: Reflect the octant $x \geq y \geq 0$ across the line $x = y$ and then reflect again across the y-axis. The image of the point $(9, 2)$ under the two reflections is

$$(9, 2) \mapsto (2, 9) \mapsto (-2, 9).$$

Any path in the octant meeting $x = y$ and the y-axis from $(6, 5)$ to $(9, 2)$ can be identified with a path directly from $(6, 5)$ to $(-2, 9)$ as in the diagram below.



hence the shortest path is the straight line from $(12, 10)$ to $(-3, 18)$ which has length

$$L = \sqrt{(12 + 3)^2 + (10 - 18)^2} = \sqrt{15^2 + 8^2} = 17$$

6. What is the maximum possible area of a right angle triangle with the property that the sum of its side lengths, i.e. its perimeter, is equal to the sum of the squares of its side lengths?

Solution: Let the side lengths be a, b and c with $a^2 + b^2 = c^2$. To satisfy

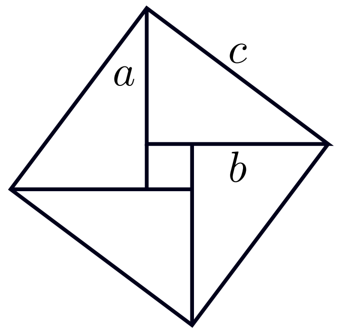
$$a + b + c = a^2 + b^2 + c^2 \tag{3}$$

we cannot have all side lengths ≤ 1 hence $c > 1$, and we will see that c is bounded. Define the shape, or similarity class, of a right angle triangle by the angle $0 < \theta < \pi/2$ so that $a = c \cos \theta$ and $b = c \sin \theta$. Note that any similarity class of right angle triangles can satisfy (3) after rescaling

$(a,b,c) \mapsto (\lambda a, \lambda b, \lambda c)$ and this produces a unique triangle up to congruence in each similarity class. Thus we can write the hypotenuse and area as functions of θ, given by $c(\theta)$ and $A(\theta)$. Then

$$\begin{aligned} a+b+c = a^2+b^2+c^2 &\Rightarrow c(\cos\theta + \sin\theta + 1) = c^2(\cos^2\theta + \sin^2\theta + 1) = 2c^2 \\ &\Rightarrow c(\theta) = \tfrac{1}{2}(\cos\theta + \sin\theta + 1) \end{aligned}$$

and this achieves a maximum when $\theta = \pi/4$ (since the unit circle is tangent to the line $x + y = \sqrt{2}$).



The area of the triangle is $A = \frac{1}{2}ab$ and from the diagram we see that $4A \leq c^2$ with equality only for the isosceles triangle. Hence, for any θ,

$$4A(\theta) \leq c(\theta)^2 \leq c(\pi/4)^2 = 4A(\pi/4)$$

Thus the maximum area is achieved when $\theta = \pi/4$ and it is given by

$$A(\pi/4) = \tfrac{1}{4}c(\pi/4)^2 = \tfrac{1}{4}\left(\tfrac{1}{2}(\sqrt{2}+1)\right)^2 = \frac{3+2\sqrt{2}}{16}.$$

7. A set of three distinct positive integers $\{a,b,c\}$ is defined to be a *tame triple* if $a+b+c = 17$. Define a set S of positive integers to be *tame* if each element of S is contained in *exactly two* tame triples that are subsets of S. For example, the set $\{1,2,3,...,17\}$ is not tame since 1 is contained in more than two tame triples $\{1,2,14\}, \{1,3,13\}, \{1,4,12\}, ...$ (and 17 is not contained in a tame triple). How many elements must a non-empty tame set S contain?

Solution: Let $|S|$ equal the number of elements in S. Take the $2|S|$ integers consisting of two copies of S, so each element appears twice, and partition this into distinct tame triples. Hence $2|S|$ is divisible by 3 hence $|S|$ is divisible by 3. The sum of the $2|S|$ integers is equal to $17 \times 2|S|/3$ hence the sum of the elements of S is $17/3 \times |S|$.

If $|S| = 12$ and $S = \{a_1, ..., a_{12}\}$ then

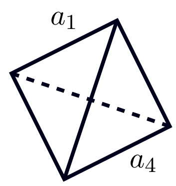
$$a_1 + ... + a_{12} = 17/3 \times 12 = 68$$

but the minimum sum of 12 distinct positive integers is $1+2+...+12 = 78 > 68$ which is a contradiction. The same argument applies to $|S| = 15$, hence $|S| < 12$.

Now $|S| \neq 3$ since S must contain at least two different tame triples. Thus $|S| = 6$ or $|S| = 9$.

We can place the elements of S on the edges of triangles glued together as in the diagram. The triangles give the tame triples and each edge is incident to exactly two triangles, since each element of S is contained in exactly two tame triples, so we glue along the edge.

If $|S| = 6$, choose an integer in S, then it defines two tame triples represented by two adjoining triangles, given by the solid edges in the diagram. The final, dotted, edge can be placed to form a tetrahedron, possibly after swapping two values within a triangle to ensure they match with their tame triples. Each integer of $S = \{a_1, ..., a_6\}$ is placed on an edge although only a_1 and a_4 are shown.



The sum around any face is 17. Assign a plus sign to the two faces with edge a_1 and a minus sign to the two faces with edge a_4. Take the alternating sum of the faces according to these signs to get

$$0 = 17 + 17 - 17 - 17 = 2a_1 - 2a_4$$

to conclude equality $a_1 = a_4$, which is a contradiction. Hence $|S| \neq 6$.

Thus we must have $|S| = 9$. In this case,

$$a_1 + ... + a_9 = 17/3 \times 9 = 51$$

which can be solved by

$$S = \{1, 2, 3, 4, 5, 7, 8, 9, 12\}.$$